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Triangle Formation Design in Eccentric Orbits Using Pseudospectral Optimal Control

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In this paper, we further a general framework for the formation design of distributed space systems. Based on constrained nonlinear optimal control theory, the formation of multi-spacecraft is designed by pseudospectral methods. The approach deals with the full nonlinear dynamics without using any linearization/approximation techniques; and is applicable to elliptical reference orbits. The method is tested on various formation problems including equilateral triangular formation over eccentric orbits. This framework provides a unified and straightforward way to design large-scale formation for distributed space systems.

I. Introduction

A distributed space system (DSS) is a multi-agent system of systems that has long been recognized[1] as a key technology area to enhance the scope of both military and civilian space applications. For example, a system of three spacecrafts (see Fig. 1) orbiting in formation can provide a very long baseline for the

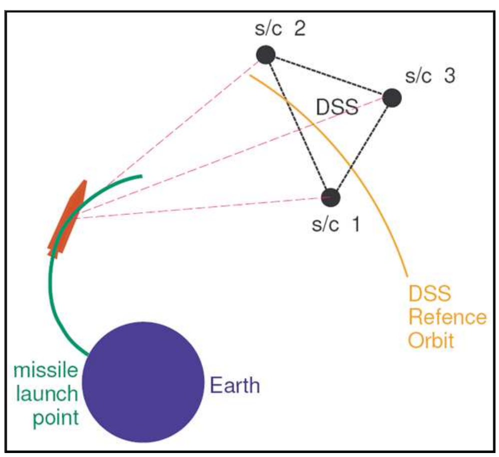

Figure 1. **Example of a DSS providing a long baseline for precision missile tracking.**

precise detection and tracking of a missile launch. Kilometer-size base lengths cannot be provided by a

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single monolithic spacecraft, and hence the DSS shown in Fig. 1 has the capability to substantially enhance our nation's missile defense capability. Many other DSS designs are possible for a great many Intelligence, Surveillance and Reconnaissance (ISR) capabilities as described in a USAF report[1] over a decade ago.

In principle, a space formation can be achieved easily by dynamic inversion and precision control; however, the propellant consumption for this control scheme could be so extraordinarily expensive that it would lead to the erroneous conclusion that a DSS is not viable from an engineering viewpoint. For many ISR applications, it is not necessary for a DSS to be a rigid formation.[2–4] This is because the ISR requirements do not specify rigidity, but that the relative orbits be almost periodic. These formation requirements can be described in terms of state variable constraints and almost periodic functions. In the case of a controlled formation, it is desirable to maintain the formation with minimum fuel. Given the orbit constraints and the minimum fuel requirements, in this paper we explore allowable formation configurations as a problem of nonlinear optimal control with state-control constraints; and apply the methods on equilateral triangle formation design.

Most of the existing results on formation design are based on the analysis of the relative motion between two satellites. When the reference orbit is circular, the differential gravity field can be directly linearized resulting in the well known Hill-Clohessy-Wiltshire (HCW) equations. The relative motion of the spacecraft can be analytically obtained from HCW equations. Over the last decade, many methods have been proposed for analyzing the nonlinear effects in the relative dynamics and the drift induced by the eccentricity of the reference orbits.[5–8] These analytic design methods are usually based on linearization or high-order approximations of the relative dynamics to reduce the nonlinear effects. When one allows eccentric reference orbits and includes the J_2 effects, the relative motion requires more involved analysis. In some cases, J_2 invariant orbits can be used to negate the differential perigee rotation or the differential nodal precession to design the formation.[9] While these analytic design methods have achieved great success in the design of the formation for a two spacecraft system, extending the results to multi-spacecraft is not always easy or straightforward.

In this paper, our objective is to provide a unified framework for the formation design of distributed space systems. As first proposed by Ross et al,[2,4] the formation problem can be formulated as a constrained nonlinear optimal control problem. Geometry requirements on the formation are modeled as path constrains. The approach deals with the full nonlinear dynamics directly without using any linearization/approximation techniques; therefore, it allows eccentric reference orbits and large relative distances between spacecraft. The proposed method is also scalable to distributed space systems that consist of a large number of spacecraft. In this paper, the J_2 perturbation is not considered. However, the proposed approach has the potential to deal with high-order perturbations. As demonstrated on various formation design problems, including the equilateral triangular formation in elliptical reference orbits, the proposed approach can be applied to a wide range of problems in a straightforward manner. The core algorithm remains the same for different requirements on the shape of the formation, the number of agents and the nature of the reference orbits. Such portability is especially useful in the analysis and design of real-world satellite formation applications.

The resulting optimal control problem is solved using pseudospectral (PS) methods. Over the last decade, PS methods for optimal control have moved rapidly from theory to practice to flight application. The recent application of PS optimal control onboard the International Space Station[10] marks one of the many milestones in the recent developments. One particular advantage of PS optimal control is that it offers spectral convergence rate if the solution is smooth. For formation design, the trajectory is naturally smooth. Therefore, the fast convergence rate of PS methods allows it to accurately capture the nonlinear effects with a relative small number of discretization nodes.

II. Formation Design Through Constrained Optimal Control

Given the orbit of a chief spacecraft, what we are interested in formation design is to find the orbit of the deputies, so that the chief and deputies can maintain certain geometric structures for a long period of time and preferably without using any propellant. In the orbital design for satellite formations, we face several challenges:

- nonlinearity in the dynamics,
- fuel-optimality for feasibility,
- higher-order effects such as J_2; and

- the portability of design methods for various formation requirements and different numbers of spacecraft.

In this paper, we focus on the issues of formation design with full nonlinear dynamics and portability to multi-spacecraft. The proposed approach is aimed at answering these challenges by developing a formation design method that takes into consideration the full nonlinear dynamics. The method is portable to formations with different boundedness requirements and/or different number of spacecrafts.

A typical optimal control problem can be summarized as

$$\begin{aligned} \text{Minimize} \quad J[x(\cdot),u(\cdot)] &= \frac{1}{2}\int_{t_0}^{t_f} F(x(t),u(t))dt + E(x(t_0),x(t_f)) \\ \text{Subject to} \quad \dot{x}(t) &= f(x(t),u(t)) \\ e(x(t_0),x(t_f)) &= 0 \\ h(x(t),u(t)) &\leq 0 \end{aligned}$$

where $x \in \mathbb{R}^{N_x}, u \in \mathbb{R}^{N_u}, F : \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \to \mathbb{R}, E : \mathbb{R}^{N_x} \times \mathbb{R}^{N_x} \to \mathbb{R}, f : \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \to \mathbb{R}^{N_x}, e : \mathbb{R}^{N_x} \times \mathbb{R}^{N_x} \to \mathbb{R}^{N_e}, h : \mathbb{R}^{N_x} \times \mathbb{R}^{N_u} \to \mathbb{R}^{N_h}$.

To formulate the formation design problem as an optimal control problem, we need to determine three things: the dynamical constraints, boundary/path constraints and cost function. In this paper, the dynamics of the spacecraft is modeled by

$$\begin{aligned} \dot{x}_i &= v_{xi} \\ \dot{y}_i &= v_{yi} \\ \dot{z}_i &= v_{zi} \\ \dot{v}_{xi} &= -\frac{\mu}{r_i^3}x_i + T_{xi} \\ \dot{v}_{yi} &= -\frac{\mu}{r_i^3}y_i + T_{yi} \\ \dot{v}_{zi} &= -\frac{\mu}{r_i^3}z_i + T_{zi}, \end{aligned} \tag{1}$$

where $r_i = \sqrt{x_i^2 + y_i^2 + z_i^2}$, μ is the earth gravitational constant, and subscript $i = 1, 2, \cdots$ denotes the i-th spacecraft in the formation. $(T_{xi}(t), T_{yi}(t), T_{zi}(t))$ is the control on the i-th spacecraft. Throughout the paper, $i = 1$ denotes the chief spacecraft, whose trajectory is given. Note that, we are not using the relative dynamics or linearizations or high-order approximations. This approach allows us to deal with the nonlinearities in a direct and straightforward manner. Scaling to multi-body formations is also made easy. In (1), no high-order perturbations, such as J_2, are included. While, it is possible to include such high-order perturbations into the optimal control formulation for formation design, we will not consider it in this paper. The topic is currently under investigation.

To minimize the deviation of the formation over long periods, we impose the following periodic boundary conditions.

$$\mathbb{X}_i(0) = \mathbb{X}_i(T), \tag{2}$$

where $\mathbb{X}_i(t) = (x_i, y_i, z_i, v_{xi}, v_{yi}, v_{zi})(t)$, the states of the i-th spacecraft. T is the period of the given chief spacecraft or reference orbit. With this boundary condition, we only need to consider the trajectories within the time span $t \in [0, T]$; and these trajectories can be extended to many periods due to the periodicity.

To maintain certain geometry shape of the formation, path constraints need to be introduced. A specific geometry path-constraint is imposed on the relative distance between different spacecraft.

$$D^L \leq D_{ij}(t) = \sqrt{(x_i(t)-x_j(t))^2 + (y_i(t)-y_j(t))^2 + (z_i(t)-z_j(t))^2} \leq D^U \tag{3}$$

for all $t \in [0, T]$. Lower bound $D^L > 0$ is for collision avoidance and upper bound $D^U > 0$ is to keep a bounded formation. This constraint allows the formations with variable relative distance. In general, especially in the case of elliptical orbits, a variable relative distance facilitates formations with zero or minimum propellant cost. Other geometry constraints can be imposed on the inertial frame position (x_i, y_i, z_i) also. For instance, to design a projected circular formation, one can transfer the relative position into LVLH (Local Vertical Local Horizontal) frame and enforce the projected relative distance to be constant.[3]

The choice of cost function provides extra freedom in the design of formation. For instance, when propellant is expended to maintain a specific formation, one can incorporate thrust into the cost function and minimize certain norm of the control effort as

$$\text{Minimize} \quad J[x(\cdot), u(\cdot)] \quad = \quad \int_0^T \sum_{i\geq 2} ||(T_{xi}(t), T_{yi}(t), T_{zi}(t))||\, dt$$

where $(T_{xi}(t), T_{yi}(t), T_{zi}(t))$ is the control on i-th spacecraft. If a natural formation exists, the cost function can be the designer's choice to improve certain performance of the trajectory.[3] For example, one can try to minimize the distortion of the formation as we will show later in the design of equilateral triangular formations.

Once the dynamical constraints, path/boundary constraints and cost function are determined, the formation design problem becomes a standard optimal control problem. As mentioned before, our goal is to develop a method of formation design that is portable to different formation requirements and different number of satellites. The optimal control problem formulated in this section is clearly a general model that can be easily adopted to cope with different situations.

There are various methods to solve the resulting constrained nonlinear optimal control problem. In this paper, we focus on pseudospectral methods which are used quite routinely within the aerospace community due to the popularity of software packages such as OTIS[11] and DIDO.[12] All the examples in this paper are solved using DIDO on a Windows based PC.

III. A Quick Background on Pseudospectral Methods

Pseudospectral methods consist of a family of computational optimal control methods. They offer an easy implementation and a high convergence rate. In this section, using recently developed weighted interpolants ideas, we briefly present PS optimal control methods through a unified approach.

The key to modern computational optimal control is the approximation of the function/trajectory. Given an arbitrary function $y(t)$, in PS methods, $y(t)$ is approximated by a polynomial $y^N(t)$ as

$$y(t) \approx y^N(t) = \sum_{j=0}^{N} \frac{W(t)}{W(t_j)} \phi_j(t) y_j, \;\; a \leq t \leq b$$

where the nodes $t_j, j = 0, ..., N$ are a set of distinct interpolation nodes (defined later) on the interval $[a, b]$, the weight function $W(t)$ is a positive function on the interval, and $\phi_j(t)$ is the $Nth-$ order Lagrange interpolating polynomial that satisfies the relationship $\phi_j(t_k) = \delta_{jk}$. This implies that

$$y_j = y^N(t_j), \;\; j = 0, ...N.$$

An expression for the Lagrange polynomial can be written as[14]

$$\phi_j(t) = \frac{g_N(t)}{g_N'(t_j)(t - t_j)}, \;\; g_N(t) = \prod_{j=0}^{N}(t - t_j).$$

One important tenant of PS approximation of functions is that differentiation of the approximated functions can be performed by differentiation of the interpolating polynomial,

$$\frac{dy^N(t)}{dt} = \sum_{j=0}^{N} \frac{y_j}{W(t_j)} [W'(t)\phi_j(t) + W(t)\phi_j']$$

Since only the values of the derivative at the nodes t_i are required for PS methods, then we have,

$$\left.\frac{dy^N(t)}{dt}\right|_{t_i} = \sum_{j=0}^{N} \frac{y_j}{W(t_j)} [W'(t_i)\delta_{ij} + W(t_i)D_{ij}] = \sum_{j=0}^{N} D_{ij}[W] y_j$$

where we use $D_{ij}[W]$ as a shorthand notation for the W-weighted differentiation matrix,

$$D_{ij}[W] := \frac{[W'(t_i)\delta_{ij} + W(t_i)D_{ij}]}{W(t_j)}$$

and D_{ij} is usual unweighted differentiation matrix given by,

$$D_{ij} := \left.\frac{d\phi_j(t)}{dt}\right|_{t=t_i}$$

Thus, when $W(t) = 1$, we have

$$D_{ij}[1] = D_{ij}$$

From Eq. (4), the unweighted differentiation matrix, $D_{ij} = \phi_j'(t_i)$, has the form,

$$D_{ij} = \begin{cases} \dfrac{g_N'(t_i)}{g_N'(t_j)} \dfrac{1}{(t_i - t_j)}, & i \neq j \\ \\ \dfrac{g_N''(t_i)}{2g_N'(t_i)}, & i = j \end{cases}$$

The above equations are the general representations of the derivative of the Lagrange polynomials evaluated at arbitrary interpolation nodes. Thanks to Runge, it is well-known that an improper selection of the grid points can lead to disastrous consequences. In fact, a uniform distribution of grid points is the worst possible choice for polynomial interpolation and hence differentiation. On the other hand, the best possible choice of grid points for integration, differentiation and interpolation of functions are Gaussian quadrature points. Consequently, all PS methods use Gaussian quadrature points.

Let $\{P_N(t)\}$ be a sequence of polynomials orthogonal with respect to an appropriate inner product; and let $t_0 = -1 < t_1 < \cdots < t_N = 1$ be the nodes. There are three types of Gaussian quadrature points commonly used in solving optimal control problems:

zeros of $P_{N+1}(t)$	Gauss quadrature nodes, or
the end points and the critical points of $P_{N+1}(t)$	Gauss-Lobatto nodes, or
the left end point and the zeros of $P_{N+1}(t) - \frac{P_{N+1}(-1)}{P_N(-1)} P_N(t)$	Gauss-Radau nodes.

Fig.2 illustrates the distribution of these quadrature nodes. One distinctive feature of these nodes is the

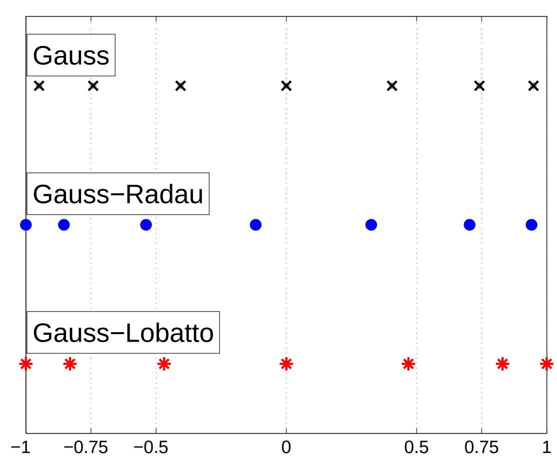

Figure 2. Illustration of quadrature points.

nonuniform distribution. The nodes are much more dense towards the end points. Indeed, the distance between the nodes converges at a rate of N^{-2} around end points[14,15] compared to N^{-1} of uniform distribution. This property effectively prevent Runge phenomenon.[14,15]

Let $\bar{x}_k^N$ and $\bar{u}_k^N$ be an approximation of a feasible solution $(x(t), u(t))$ evaluated at the node t_k. Then, as a result of the discretization, the optimal control problem is transformed to a finite dimensional constrained nonlinear optimization problem. In the case of a Legendre PS method, this problem can be written as,

Problem $\mathbf{B^N}$: Find $\bar{x}_k^N$ and $\bar{u}_k^N$, $k = 0, 1, \ldots, N$, that minimize

$$\bar{J}^N(\bar{X}, \bar{U}) = \sum_{k=0}^{N} F(\bar{x}_k^N, \bar{u}_k^N) w_k + E(\bar{x}_0^N, \bar{x}_N^N)$$

subject to

$$\begin{aligned} \left\| D \begin{pmatrix} \bar{x}_{i0}^N \\ \vdots \\ \bar{x}_{iN}^N \end{pmatrix} - \begin{pmatrix} f_i(\bar{x}_0^N, \bar{u}_0^N) \\ \vdots \\ f_i(\bar{x}_N^N, \bar{u}_N^N) \end{pmatrix} \right\|_\infty &\le \delta_{1N} \mathbf{1} \quad i = 1, 2, \ldots, r-1 \\ h(\bar{x}_k^N, \bar{u}_k^N) &\le \delta_{2N} \cdot \mathbf{1}, \\ \left\| e(\bar{x}_0^N, \bar{x}_N^N) \right\|_\infty &\le \delta_{3N} \end{aligned}$$

for all $0 \le k \le N$. δ_{iN} is a given small number representing the discretization tolerance.

The theoretical analysis in Ref. [17–19] shows the well-posedness of PS discretization, i.e., preserving the feasibility of the original continuous problem and it is a consistent approximation[20] to the original optimal control problem. Problem B^N is then solved by a spectral algorithm[16] that utilizes a sequential quadratic programming approach. This algorithm is implemented in the software package DIDO.[12]

IV. Some Computational Issues

Canonical units are frequently used to scale the orbit dynamical equations.[21–23] In this paper, we adopt the following units which provide a good balance for position and velocity.

$$\begin{aligned} t &= \sqrt{\frac{r_0^3}{\mu}} \tau; \quad x_i = r_0 \bar{x}_i; \quad y_i = r_0 \bar{y}_i; \quad z_i = r_0 \bar{z}_i; \\ v_{xi} &= \sqrt{\frac{\mu}{r_0}} \bar{v}_{xi}; \quad v_{y_i} = \sqrt{\frac{\mu}{r_0}} \bar{v}_{yi}; \quad v_{zi} = \sqrt{\frac{\mu}{r_0}} \bar{v}_{zi}; \\ T_{xi} &= \frac{\mu}{r_0^2} \bar{T}_{xi}; \quad T_{yi} = \frac{\mu}{r_0^2} \bar{T}_{yi}; \quad T_{zi} = \frac{\mu}{r_0^2} \bar{T}_{zi} \end{aligned}$$

where r_0 can be any constant. In the simulation we choose r_0 to be the semi-major axis of the reference orbit. Under this set of units the dynamical equations are changed to

$$\begin{aligned} \dot{\bar{x}}_i &= \bar{v}_{xi} \\ \dot{\bar{y}}_i &= \bar{v}_{yi} \\ \dot{\bar{z}}_i &= \bar{v}_{zi} \\ \dot{\bar{v}}_{xi} &= -\frac{1}{\bar{r}_i^3} \bar{x}_i + \bar{T}_{xi} \\ \dot{\bar{v}}_{yi} &= -\frac{1}{\bar{r}_i^3} \bar{y}_i + \bar{T}_{yi} \\ \dot{\bar{v}}_{zi} &= -\frac{1}{\bar{r}_i^3} \bar{z}_i + \bar{T}_{zi} \end{aligned}$$

where $\bar{r}_i = \sqrt{\bar{x}_i^2 + \bar{y}_i^2 + \bar{z}_i^2}$.

In addition to the scaling of the unit, we also adopt other numerical techniques to avoid ill-scaled constraints. For example, while the dynamics is re-scaled, the constraints on the relative distance are imposed in their original units. This is because the relative distance between satellites is a number that is much smaller relative to the radius of the orbits. If the same scale is used, then the bounds of the relative distance would easily exceed the machine and software precision limit.

In all the computations reported in this paper, we impose the following bounds on the positions of the deputies.

$$
\begin{array}{rcccll}
D - D^U & \leq & x_i(t) - x_1(t) & \leq & D^U - D, & i \geq 2 \\
D - D^U & \leq & y_i(t) - z_1(t) & \leq & D^U - D, & i \geq 2 \\
D - D^U & \leq & z_i(t) - z_1(t) & \leq & D^U - D, & i \geq 2
\end{array}
$$

where D is the desired relative distance, D^U is the upper bound on the relative distance introduced in (3) and $(x_1(t), y_1(t), z_1(t))$ is the given reference orbit. This search region includes all feasible trajectories with relative distance satisfying path constraint (3).

V. A Simple Two Body Formation Design Problem

For an easy demonstration and clear explanation of the proposed optimal control approach of formation design, we apply the method to a simple two body formation design problem. That is, given the trajectory of the chief

$$(x_1(t), y_1(t), z_1(t), v_{x1}(t), v_{y1}(t), v_{z1}(t))$$

over one period $[0, T]$, we want to find the trajectory of the deputy so that the relative distance remains inside a given bound for a long period of time. This problem has been well studied in the literature using analytical techniques. For example, if the dynamic model assumes a circular reference orbit and a spherical Earth then one can use the Clohessy-Wiltshire equations solution to design the desired formation. In this case, it is well known that a) to avoid secular drift the semi-major axes of the satellites need to be equal, b) the size of the 2x1 ellipse is defined by the differential eccentricity, and c) the magnitude of the out-of-plane motion is defined by the differential inclination and right ascension. When eccentric reference orbits are considered, various high-order approximations of the relative dynamics have been proposed to analytically design the formation.[5–8] When the J_2 perturbation is included, a rich literature is available for analytic formation design. For instance, J_2 invariant orbits can be used to negate the differential perigee rotation or the differential nodal precession. Numerical methods have also been used to solve two spacecraft formation designs.[9] For example, in Ref.[2,3], the Legendre pseudospectral method was applied on a simplified relative dynamics and a two-body formation is designed for both circular and elliptical reference orbit. In Ref.[4], pseudospectral methods are used to design spacecraft formations around libration point. Such a rich history on this problem provides a good reference to compare the proposed approach.

Our optimal control formulation for two spacecraft formation is summarized as

$$\text{Minimize} \quad J[x(\cdot), u(\cdot)] \;=\; \int_0^T T_{x2}^2(t) + T_{y2}^2(t) + T_{z2}^2(t)dt$$

subject to state dynamics (1), periodic boundary condition (2) and relative distance constraint (3). The cost function is chosen to be the quadratic function of controls. It serves as a negative test to verify the solution. If the cost is non zero, it indicates the solution is wrong since a natural formation exists. Indeed, in all the results we obtained, the costs are zeros.

To test the proposed methods, we choose a reference orbit with 10,000km semi-major axis and 45° inclination. The eccentricity is set to be zero first, i.e., circular reference orbit. The reference trajectory is generated using MATLAB build in Runge-Kutta propagation function RK45. The formation distance between the chief and the deputy is 50km. In the relative distance path constraint (3), the lower and upper bounds are set to be $D^L = 50km - 0.05km$ and $D^U = 50km + 0.05km$. In other words, we want the relative distance to be 50km with 0.1% of accuracy.

A 64 node solution for the aforementioned setting is demonstrated in the following. First, the resulting trajectory is propagated into nonlinear dynamics (1) for 10 periods and plotted out in the LVLH frame in Fig.3. As apparent from Fig.3, the two spacecraft are able to maintain a stable formation for a long period to time. This is further demonstrated in Fig.4 where the relative distance is plotted.

To show the difference from a linearization approach, consider the following linear system derived for the

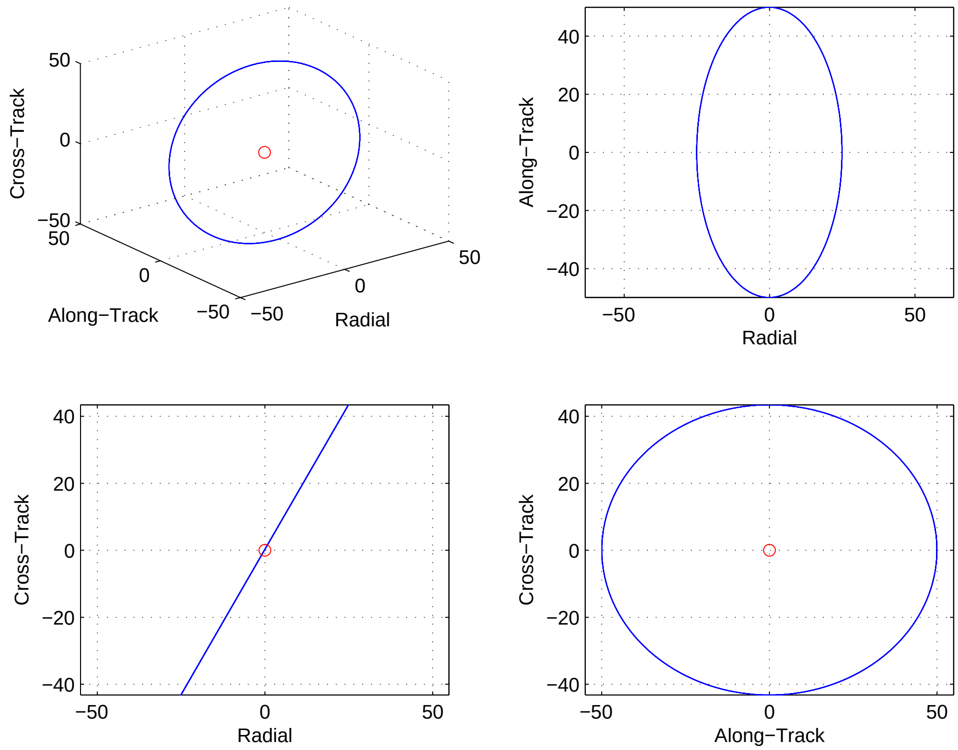

Figure 3. **Relative position (in LVLH frame) between spacecraft 1&2 in 10 periods. Red circle represents the position of Chief.**

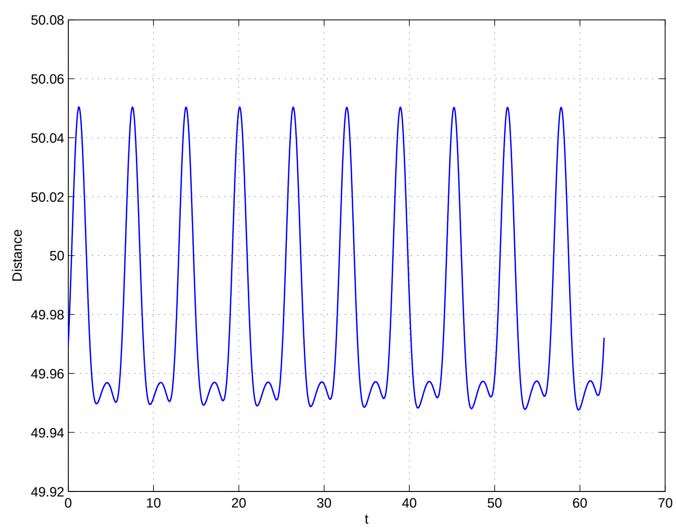

Figure 4. **Relative distance in 10 periods.**

dynamics of the relative motion in LVLH frame.

$$\begin{pmatrix} \ddot{e}_x \\ \ddot{e}_y \\ \ddot{e}_z \end{pmatrix} = \begin{pmatrix} 0 & 2\dot{\theta} & 0 \\ -2\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{pmatrix} + \begin{pmatrix} \dot{\theta}^2 + 2k & \ddot{\theta} & 0 \\ -\ddot{\theta} & \dot{\theta}^2 - k & 0 \\ 0 & 0 & -k \end{pmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} \tag{4}$$

where θ is the true anomaly and

$$k = \frac{\mu}{r^3}$$

with r being the radius of the chief satellite. The state (r, θ) is subject to the following dynamics

$$\begin{aligned}\ddot{r} &= r\dot{\theta}^2 - \frac{\mu}{r^2}\\ \ddot{\theta} &= -2\frac{\dot{r}}{r}\dot{\theta}\end{aligned}$$

Equation (4) is a time-varying linear system which has been widely used in solving formation problems. In an elliptical orbit, the nonlinear part becomes time-varying which makes the problem much more complicated.

The previous formation computed using the full nonlinear dynamics cannot be captured by using the linearization, as shown by the following simulation. Propagating the same initial condition obtained using the optimal control technique to the linearized system (4); the result is shown in Fig.5. After about five periods, the formation deteriorates in contrast to the well maintained configuration under the full nonlinear dynamics as shown in Fig. 3. This indicates that the formation computed by using the full nonlinear model *cannot* be accurately captured by solving linearized dynamics. While there are methods to improve linearization techniques to achieve better approximation, the limitations of linearization are fundamental. For formations with large relative distances, linearization may be inaccurate.[13]

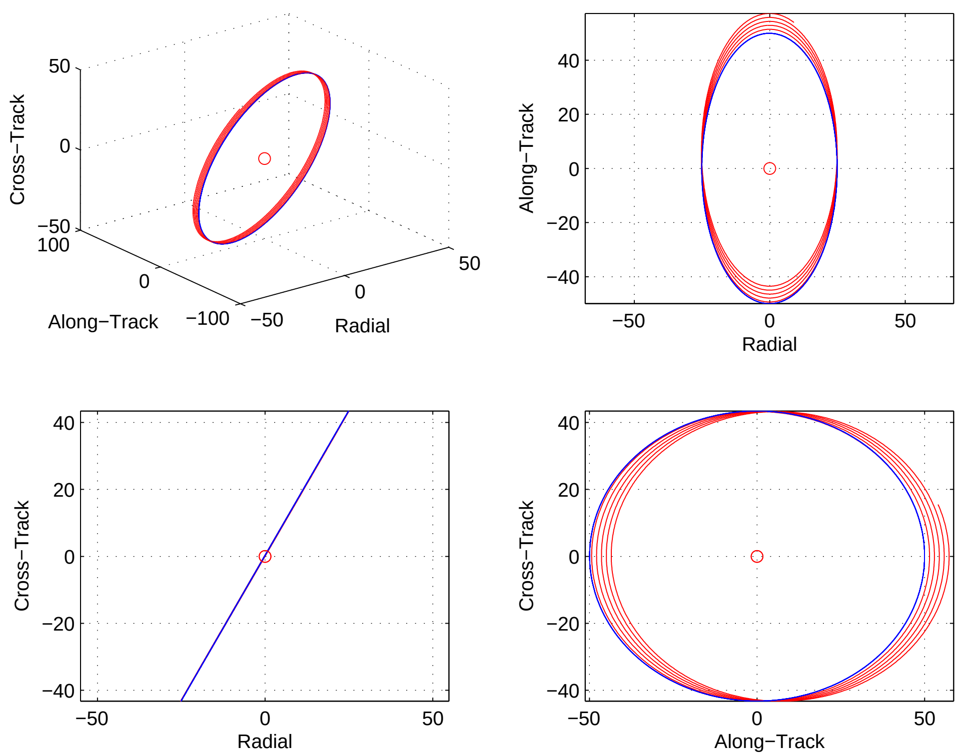

Figure 5. Relative position between spacecraft 1&2 in 5 periods. The red line is the trajectory by propagating the initial condition to dynamic (4); blue line is the trajectory using full nonlinear dynamic.

If the leading spacecraft is along an elliptical orbit, the nonlinearity in the problem has significant impact on the formation. Analytic formation designs usually are much more involved than circular reference orbit. However, in PS optimal control, changing the reference orbit is straightforward. For example, if the chief spacecraft is on an eccentric orbit, we can simply adopt the same program and change the reference trajectory of the chief. Of course, in this case, the tolerance on the relative distance needs to be adjusted since no exact circular relative orbit is expected (without any control).

For example, we apply PS optimal control on a reference orbit with an eccentricity of 0.5, a semi-major axis of 43,053km and an inclination of 45°. The relative distance constraint is set to be

$$45km \quad \leq D(t)\ 55km$$

The solution propagated in 10 periods is shown in Fig.6.

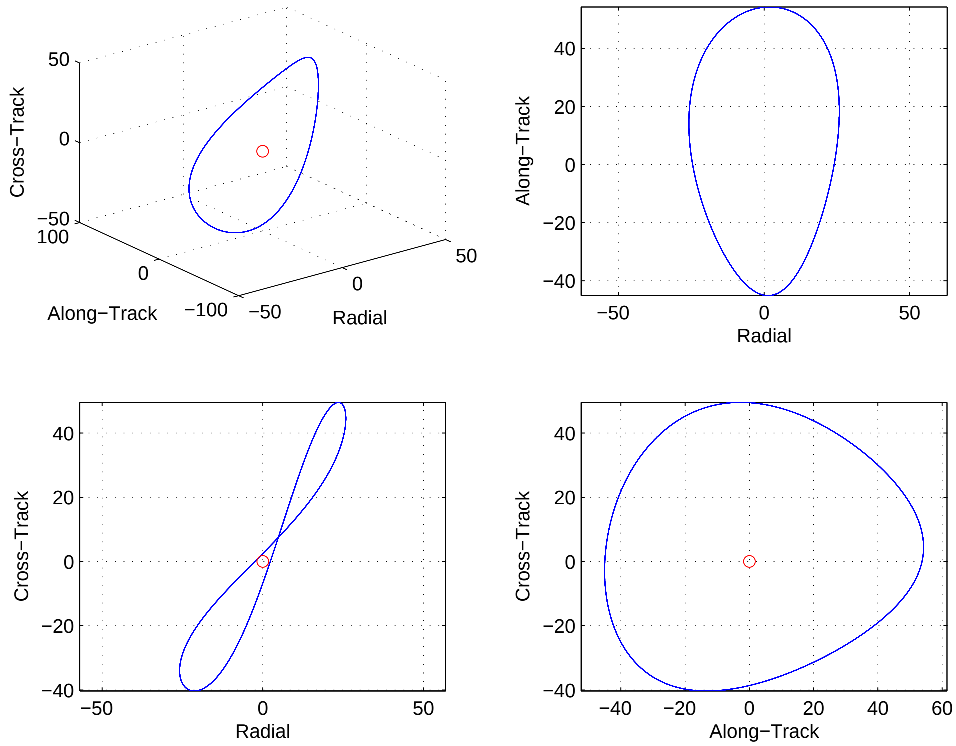

Figure 6. Relative position (in LVLH frame) between spacecraft 1&2 in 10 periods. Red circle represents the position of Chief which is on an elliptical orbit with 0.5 eccentricity.

There are multiple solutions to the proposed optimal control problem. By using different initial guesses, we can easily find other trajectories satisfying the same set of constraints. For the purpose of brevity, these solutions are not shown here. The none-unique nature of the optimal solution gives us the freedom to impose additional constrains to facilitate the design requirements.

VI. Equilateral Triangle Formation

One particular advantage of optimal control based formation design is its scalability. While two-body formations can be solved using various analytic design methods, extending these results to multi-spacecraft is not always easy or straightforward. However, with this optimal control approach, adding more agents increases the size of the optimization problem but does not change the fundamental problem formulation. Recent advances in PS optimal control algorithms reveal that solving a large-scale optimal control problem is well within present-day computational capabilities. In this section, we show how to apply this optimal control approach to design an equilateral or near equilateral triangle formation.

Denote $\mathbb{X}_i(t) = (x_i(t), y_i(t), z_i(t), v_{xi}(t), v_{yi}(t), v_{zi}(t))$ as the state of the i-th spacecraft. Given the trajectory of the chief, $\mathbb{X}_1(t)$, determine the states of the deputies, $\mathbb{X}_i(t)$, $i = 2, 3$, so that the nonlinear dynamics (1) is satisfied;

- periodic end-point condition: $\mathbb{X}_i(0) = \mathbb{X}_i(T)$ is satisfied, where T is the period of the chief;
- relative distance constraints: $D^L \leq D_{ij}(t) \leq D^U$ are satisfied, where D_{ij} is the relative distance between spacecraft i and j, and D^L, D^U are lower and upper bounds allowed;
- and the following cost function is minimized

$$J[\mathbb{X}_1(\cdot), \mathbb{X}_2(\cdot)] = \int_0^T (D_{12}(t) - D_{13}(t))^2 + (D_{12}(t) - D_{23}(t))^2 + (D_{23}(t) - D_{13}(t))^2 dt$$

The cost function is chosen to minimize the distortion from an equilateral triangle. If an equilateral triangle formation exists, $D_{12}(t) = D_{23}(t) = D_{13}(t)$ for all $t \in [0.T]$. Then, the minimum of the cost function should be zero. Note that, the equilateral triangle formation does not mean the relative distance will be a constant for all time. Depending on the position of the spacecraft on the orbit, the size of the equilateral triangle may be different. If the equilateral triangle formation does not exist, which might be true for an elliptical reference orbit, the value of the cost function provides a measure of the distortion. The smaller the value is, the closer the formation is to an equilateral triangle.

In the following example, we choose a reference orbit to be circular with radius 12756.4km and inclination 45°. The desired relative distance is set to be 50km. The resulting optimal control problem is solved using the Legendre pseudospectral method with 64 nodes. The relative orbits in chief's LVLH frame are plotted out in Fig.7. Note that the two deputies' orbits are almost the same; there is no visual difference in the plot. As shown in Fig.7, the formation is appeared to be an equilateral triangle, which is expected, since the

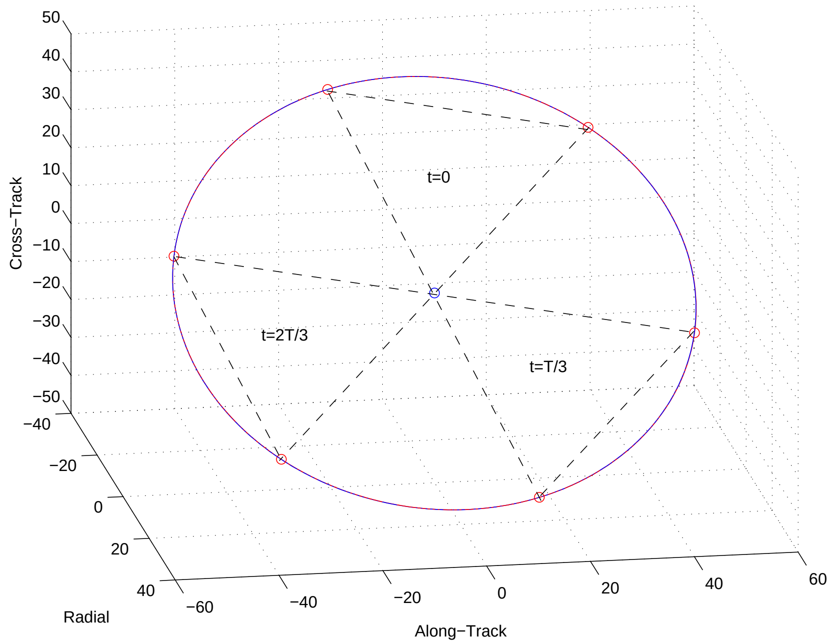

Figure 7. Equilateral triangle formation in the chief's LVLH frame.

reference orbit is circular. Indeed the relative distances between three spacecraft are all 50km with 0.1% of accuracy as shown in Fig.8.

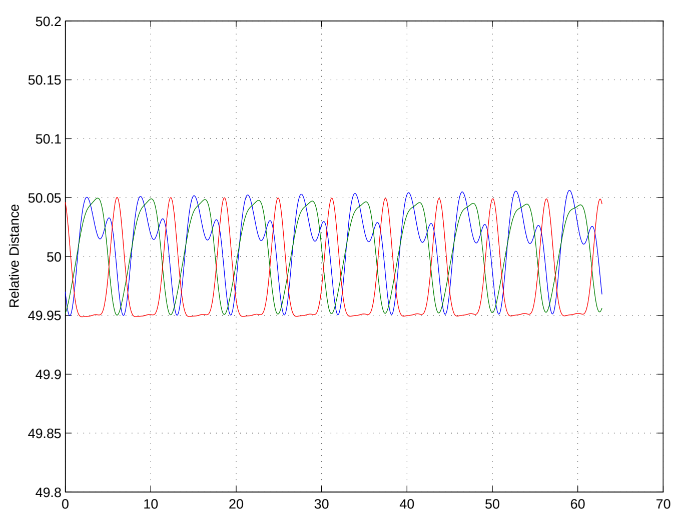

Figure 8. Relative distances between 3 spacecraft in 10 periods.

When the reference orbit is elliptic, the equilateral triangular formation may not exist. We test the example problem with different eccentricities. When $e > 0$, no equilateral triangle formation was found. The following table summarized the value of the cost function for different eccentricities. As the eccentricity

Eccentricity	Cost
0	0.00000425
0.1	0.00675924
0.2	0.00723720
0.3	0.01907360

Table 1. Cost vs Eccentricity.

increases, the cost appears to increase accordingly. As mentioned before, the chosen cost function is a measurement of the distortion of the formation away from equilateral triangle shape. The computation results indicate the distortion is monotone increasing with respect to eccentricity.

A typical solution for an elliptical reference orbit is shown in Fig.9. The orbit for the chief has an eccentricity of $e = 0.3$.

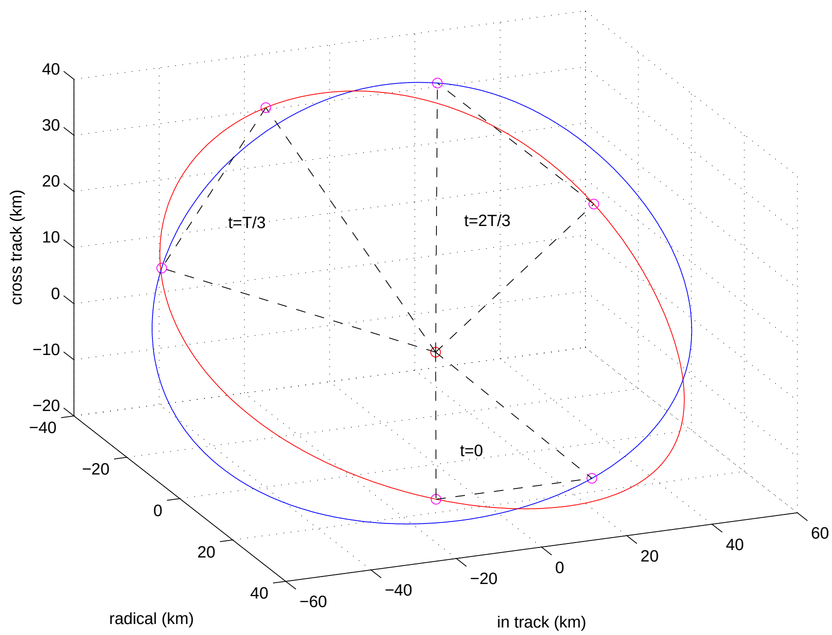

Figure 9. Demonstration of the triangle formation in the chief's LVLH frame.

There are other factors that change the shape of the formation. For instance, when the tolerance on the relative distance is relaxed, the cost appears to decrease. This is due to an expansion of the feasible set.

VII. Conclusion

In this paper, we propose a general framework for the formation design of distributed space systems. The formation design is formulated as a constrained nonlinear optimal control and solved using pseudospectral methods. The approach deals with the full nonlinear model and is applicable to elliptical reference orbits. One particular advantage offered by this approach is its scalability to multi-body formations. The technique is applied to various formation problems including the problem of equilateral triangular formation in an elliptical reference orbit.

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