Rapid Verification Method for the Trajectory Optimization of Reentry Vehicles

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Introduction

THE trajectory optimization of path-constrained nonlinear dynamic systems such as those arising in the guidance and design of reentry vehicles has long been considered a difficult problem.[1−3] Of the many methods to solve such problems, the direct collocation method with sparse nonlinear programming (NLP) has proved to be quite effective.[3] Because direct methods do not tie the resulting solutions to the Pontryagin maximum principle[4] (PMP), a two-step direct–indirect approach has been used successfully for finding extremal solutions.[5] One major drawback of the direct–indirect approach is that a significant amount of labor is necessary to derive all of the necessary conditions including complex switching and junction conditions for state-constrained problems. Even with estimates of the adjoint arcs, solving the multipoint boundary-value problem is an elaborate task.

Over the past few years, a new approach to solving complex trajectory optimization problems has been proposed.[6−9] The Legendre pseudospectral method is a particularly effective method because it provides spectrally accurate solutions for the costates and other covectors without the use of any analytical differential equations for the

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adjoints.[7,8] The software package, DIDO[9] (named after Queen Dido of Carthage), is a realization of the many advances made in solving complex trajectory optimization problems including an implementation of the symmetric covector mapping theorem (CMT).[7,8] In this Note, we focus on an application of the CMT to the verification of the first-order optimality conditions arising in the path-constrained trajectory optimization of reentry vehicles. The particular problem chosen is the standard benchmark problem of maximizing cross range.[1−3]

Overview of the Verification Method

The CMT may be articulated as follows: Given an optimal control problem, P, let P^N denote its Legendre pseudospectral approximation, where N is the order of the Legendre polynomial used in the approximation. Let P^λ denote the boundary-value problem (BVP) arising from an application of the PMP to problem P, and let $P^{N\lambda}$ denote the generalized root-finding problem obtained by applying the Karush–Kuhn–Tucker (KKT) conditions to problem P^N. The CMT asserts that if P^λ is discretized by a Legendre pseudospectral method, that is, an indirect method, then there exists an order-preserving map between these discretized covectors and the KKT multipliers associated with problem $P^{N\lambda}$. Hence, from the KKT multipliers, one can find covectors by the direct Legendre pseudospectral method as if one solved the problem by the indirect method. This is the CMT; more details are provided in Refs. 7 and 8. Clearly, one main advantage of the CMT is that it obviates the need for a two-tier direct–indirect approach and provides a quicker way to verify the optimality of the computed trajectory.

The reentry problem is solved using DIDO,[9] which implements a particular version of the Legendre pseudospectral method that exploits a sparsity pattern of the discrete Jacobian by way of the NLP solver SNOPT.[10] Feasibility is declared when the state histories obtained by a Runge–Kutta (R–K) propagation (ode45 routine in MATLAB® that implements the Dormand–Prince pair) of the initial conditions using the interpolated controls obtained from DIDO (DIDO controls) match the states obtained from DIDO (DIDO states) within a prescribed tolerance. Convergence is declared when DIDO controls match the Pontryagin-extremal controls (within some prescribed tolerance) computed through an application of the CMT (CMT controls) using the DIDO states and DIDO covectors. In the following sections, we provide details of our approach by way of the reentry problem.

Reentry Dynamic Model

The state of the vehicle is represented by the six-dimensional vector ($\boldsymbol{r}$, $\boldsymbol{\theta}$, $\boldsymbol{\phi}$, $\boldsymbol{v}$, $\boldsymbol{\psi}$, γ) consisting of radial position, longitude, latitude, speed, heading angle, and flight-path angle, respectively. The equations of motion are given by[1−3]

$$\dot{\boldsymbol{r}} = \boldsymbol{v}\sin\gamma \tag{1}$$

$$\dot{\theta} = \frac{\boldsymbol{v}\cos\gamma\cos\boldsymbol{\psi}}{\boldsymbol{r}\cos\phi} \tag{2}$$

$$\dot{\phi} = \frac{\boldsymbol{v}\cos\gamma\sin\boldsymbol{\psi}}{\boldsymbol{r}} \tag{3}$$

$$\dot{\boldsymbol{v}} = a_s - g\sin\gamma + \Omega^2\boldsymbol{r}\cos\phi(\sin\gamma\cos\phi - \cos\gamma\sin\phi\sin\boldsymbol{\psi}) \tag{4}$$

$$\dot{\boldsymbol{\psi}} = \frac{a_w}{\boldsymbol{v}\cos\gamma} - \frac{\boldsymbol{v}}{\boldsymbol{r}}\cos\gamma\cos\boldsymbol{\psi}\tan\phi - \frac{\Omega^2\boldsymbol{r}}{\boldsymbol{v}\cos\gamma}\sin\phi\cos\phi\cos\boldsymbol{\psi} + 2\Omega(\tan\gamma\cos\phi\sin\boldsymbol{\psi} - \sin\phi) \tag{5}$$

$$\dot{\gamma} = \frac{a_n - g\cos\gamma}{\boldsymbol{v}} + \frac{\boldsymbol{v}\cos\gamma}{\boldsymbol{r}} + \frac{\Omega^2\boldsymbol{r}}{\boldsymbol{v}}\cos\phi(\cos\gamma\cos\phi + \sin\gamma\sin\phi\sin\boldsymbol{\psi}) + 2\Omega\cos\phi\cos\boldsymbol{\psi} \tag{6}$$

where $g = \mu_\oplus/\boldsymbol{r}^2$ is the inverse-square gravitational acceleration with $\mu_\oplus = 0.14076539\times 10^{17}$ ft³/s² and Ω is the Earth's angular velocity, 7.2722×10^{-5} rad/s. The aerodynamic accelerations, a_s, a_n, and a_w, resolved in the tangential, normal, and binormal directions are

$$a_s = -D/m, \qquad a_n = L\cos\delta/m, \qquad a_w = L\sin\delta/m \tag{7}$$

where δ is the bank angle and $m = 7008$ slugs is the mass of the vehicle. The aerodynamic lift and drag forces are given by

$$L = q_d A C_L(\alpha), \qquad D = q_d A C_D(\alpha) \tag{8}$$

where $A = 2690$ ft² is a reference area and q_d is the dynamic pressure,

$$q_d = \tfrac{1}{2}\rho(r)v^2 \tag{9}$$

and ρ is the atmospheric density. The lift and drag coefficients are modeled as

$$C_L(\alpha) = C_{L_0} + C_{L_\alpha}\alpha \tag{10}$$

$$C_D(\alpha) = C_{D_0} + C_{D_\alpha}\alpha + C_{D_{\alpha 2}}\alpha^2 \tag{11}$$

where the coefficients $C_{L0} = -0.2070$, $C_{L_a} = 1.676$, $C_{D0} = 0.07854$, $C_{D_a} = -0.3529$, and $C_{D_{a2}} = 2.040$ are adapted from Ref. 3 with α, the angle of attack, in radians. An exponential atmospheric model

$$\rho(r) = \rho_0 e^{-\beta(r-r_0)} \tag{12}$$

is used with $\rho_0 = 0.002378$ slug/ft³, $r_0 = 20{,}902{,}900$ ft, and $\beta = 4.20168\times 10^{-5}$/ft.

Problem Definition

For an initial equatorial orbit, maximizing the cross range is equivalent to maximizing or minimizing the final latitude. Thus, the problem is to find a control history $\boldsymbol{u}(t) = [\delta(t) \quad \alpha(t)]^T$ that minimizes the final latitude

$$J = \phi(t_f) \tag{13}$$

subject to the dynamic model of the preceding section, the initial conditions, $h = 260{,}000$ ft, $v = 24{,}061$ ft/s, $\phi = 0$ deg, $\boldsymbol{\psi} = 0$ deg, $\boldsymbol{\theta} = 0$ deg, and $\boldsymbol{\gamma} = -1.064$ deg; the final conditions [term terminal area energy management (TAEM)[3] interface] $h = 80{,}000$ ft, $v = 2500$ ft/s, and $\boldsymbol{\gamma} = -5$ deg; and the heating-rate constraint $q \le 70$ Btu/ft²/s, where q is modeled as[1−3]

$$q = q_a q_r, \qquad q_a = h_0 + h_1\alpha + h_2\alpha^2 + h_3\alpha^3, \qquad q_r = C\rho^N v^M \tag{14}$$

with $N = 0.5$, $M = 3.07$, $C = 9.289\times 10^{-9}$ Btu · s$^{2.07}$/ft$^{3.57}$/slug$^{0.5}$ and

$$h_0 = 1.067, \qquad h_1 = -1.101, \qquad h_2 = 0.6988$$

$$h_3 = -0.1903$$

Note that this heating-rate model is a mixed state-control constraint[11] whose implications are discussed in the following section.

Theoretical Optimal Controls and Junction Conditions

Analytic expressions for the Pontryagin-extremal controls are easily derived by a straightforward application of the PMP.[4,11] Bank angle is given by

$$\delta = \tan^{-1}(\lambda_\psi/\lambda_\gamma\cos\gamma) \tag{15}$$

and the angle of attack must satisfy the transcendental equation,

$$-\lambda_v\left(C_{D0} + 2C_{D_{\alpha 2}}\alpha\right) + \frac{C_{L_\alpha}}{v}\left(\frac{\lambda_\psi\sin\delta}{\cos\gamma} + \lambda_\gamma\cos\delta\right) + \frac{\mu_h m q_r}{q_d A}\left(h_1 + 2h_2\alpha + 3h_3\alpha^2\right) = 0 \tag{16}$$

where λ_v, λ_ψ, and λ_γ are the costates associated with the states $\boldsymbol{v}$, ψ, and γ, respectively, and μ_h is the covector associated with the mixed state-control heating-rate constraint. If we set $q_a = 1$ so that $q = q_r$, we get a pure state constraint,

$$q = C\rho^N v^M \tag{17}$$

where $C = 7.357 \times 10^{-9}$ Btu · s$^{2.07}$/ft$^{3.57}$/slug$^{0.5}$ roughly equates the two heating rates. If this heating-rate model is chosen, then Eq (16) can be solved as

$$\alpha = \frac{C_{L_\alpha}}{2C_{D_{\alpha 2}} v \lambda_v}\left[\frac{\lambda_\psi \sin\delta}{\cos\gamma} + \lambda_\gamma \cos\delta\right] - \frac{C_{D_\alpha}}{2C_{D_{\alpha 2}}} \tag{18}$$

However, in the presence of a pure state variable constraint, the radius and velocity costates must satisfy the junction conditions,[11]

$$\lambda_r\left(t_e^-\right) = \lambda_r\left(t_e^+\right) - \eta\beta N q_e \tag{19}$$

$$\lambda_v\left(t_e^-\right) = \lambda_v\left(t_e^+\right) + \eta M q_e / v \tag{20}$$

where η is the covector associated with the implicitly imposed constraint $q_e = 70$ Btu/ft^2/s at the entry (exit) point of the heating-rate constraint. Note that, in DIDO, this point constraint is not explicitly imposed but is simply part of the inequality constraint specification, $q \leq 70$ Btu/ft^2/s. All other costates are continuous.

Results

An 80-node solution corresponding to the mixed state-control heating-rate constraint formulation is shown in Fig. 1. The solid lines represent the state histories of an R–K propagation as explained earlier. It is apparent that the results are in excellent agreement. The errors in the target states are 2.7652 ft of altitude (0.0035%), 0.4063 ft/s of velocity (0.0163%), and 0.0166 deg in flight-path angle (0.3314%). This solution was computed in 7.58 min on a 1.8-GHz, 512-MB (RAM) Pentium 4. If solution speed is desired over terminal point accuracy, a 20-node solution can be obtained in 32.75 s with target state errors of 2934 ft of altitude (3.67%), 127.6 ft/s of velocity (5.11%), and 0.45 deg in flight-path angle (8.9%). In any case, the extremality of the 80-node solution is demonstrated in Fig. 2. It is apparent that there is close agreement between the DIDO-control histories and the CMT controls calculated from Eqs. (15) and (16) using the DIDO covectors. Figure 3 shows that the Hamiltonian is constant, that is, verification of the first integral,[4] to within 10^{-3} in magnitude and equal to zero (time-free problem). Thus, the DIDO solution satisfies quite accurately the first-order necessary conditions for optimality. Note that these conditions have been verified without the tedium of indirect methods. For completion, Fig. 4 is provided to show that the constraint on the heating rate of the vehicle was maintained throughout the

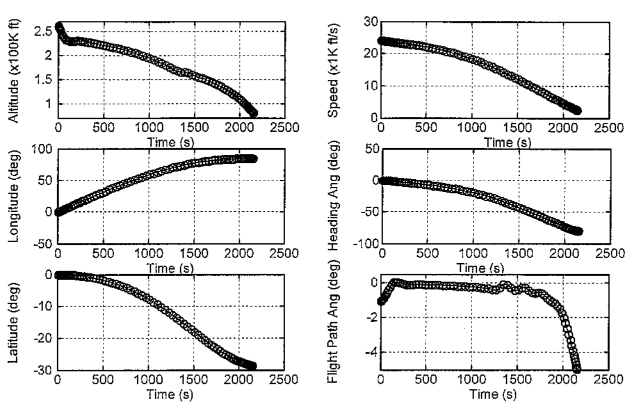

Fig. 1 State histories from a DIDO run: ——, R–K propagation and ○, DIDO solutions for 80 nodes.

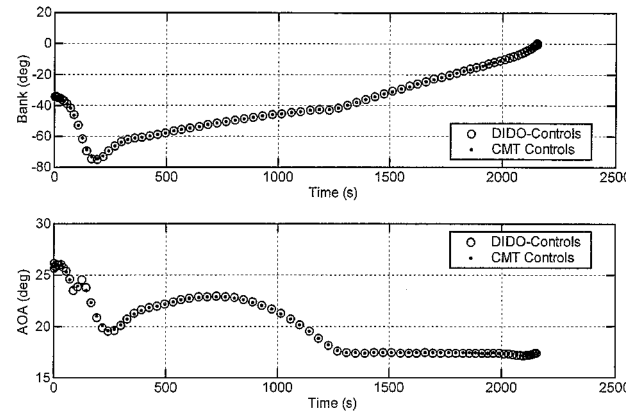

Fig. 2 Application of the CMT.

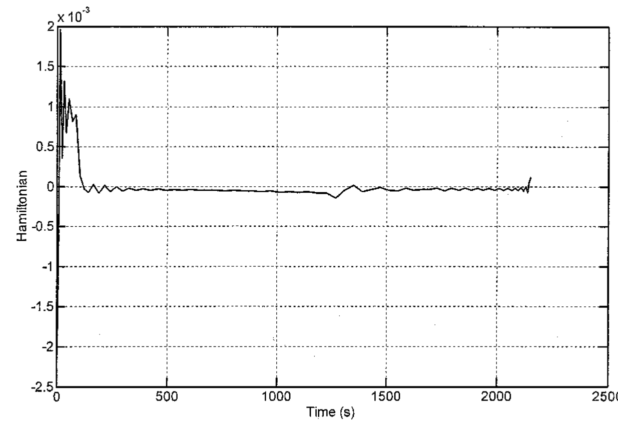

Fig. 3 First integral (time-free problem), Hamiltonian = 0.

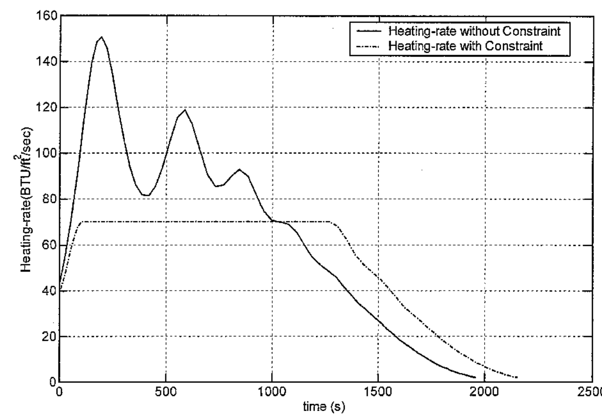

Fig. 4 Heating-rate profile with and without a heating-rate constraint.

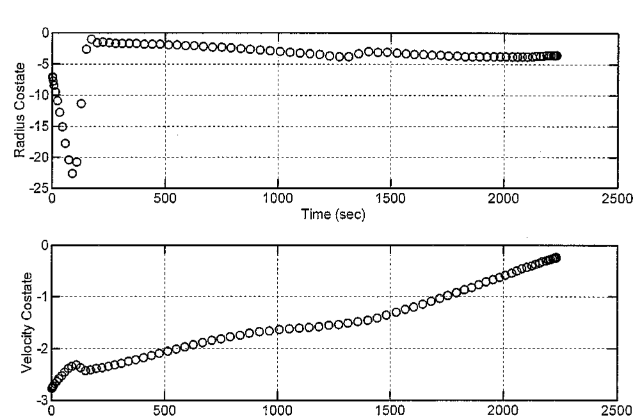

Fig. 5 Jump discontinuities in costates by way of the CMT.

trajectory. The unconstrained heating-rate profile is also shown for comparative purposes. Figure 5 shows the costates associated with radius and velocity when the heating rate is modeled as a pure state constraint [see Eq. (17)]. Note the large jump in the radius costate. When η is computed from Eq. (18) and its value is substituted in Eq. (19), the velocity costate is predicted to drop to -2.48 at (near, numerically) the entry point. The actual value of this costate is approximately -2.42, which is in close agreement with the prediction. This agreement demonstrates that the junction conditions are also satisfied reasonably accurately.

Conclusions

Accurate extremal solutions for the path-constrained reentry problem can be obtained in an efficient manner using pseudospectral methods that exploit sparsity patterns. The CMT facilitates a rapid procedure for the verification of various optimality tests. This obviates a need for the labor-intensive two-tier direct–indirect method. Computational time for the reentry problem reveals that results can be obtained within a few minutes for high accuracy and less than a minute for reduced accuracy. Further reduction in computational time is possible, for example, by way of analytic Jacobians, but it is clear that pseudospectral methods offer new possibilities in solving optimal control problems. It appears that accurate real-time solutions to complex optimal control problems are possible in the very near future. Thus, solutions to difficult problems, such as onboard trajectory generation for abort guidance, appear to be within reach.

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